On the existence of constant scalar curvature Kähler metric: a new perspective

Xiuxiong Chen In memory of Prof. Weiyue Ding June 23, 2015

1 Introduction

In 1950s, E. Calabi first proposed to study the constant scalar curvature Kähler (cscK) metric problems. His ideal is to find the best canonical metric in each given Kähler class (cf. [14, 15]), which results in a 4th order, fully nonlinear partial differential equation. When the first Chern class has a definite sign (positive, negative or zero), the cscK metric in the suitable multiple of the first Chern class reduces to a Kähler-Einstein metrics. Calabi's program on Kähler-Einstein metrics is the center of the field for the last few decades where all efforts and techniques of many mathematicians are devoted to, leading to the final resolution of this difficult problem. With the existence problem of Kähler-Einstein metric settled eventually, perhaps it is time to discuss how to attack Calabi's original problem in full generality. In this note, we propose a "new," continuity path in a given Kähler class to solve the cscK metric problem. Module out the profound difficulty in analysis, we hope this will shed light on the existence problem from direct PDE approach. This will largely be an expository paper where we concentrate on explaining various aspects of this new path, except Theorem 1.8 in which we proved openness along this proposed path. Perhaps more intriguely, while we can not prove that the K-energy functional is cocercive in terms of geodesic distance if there exists a cscK metric, we can prove that for any $t \in (0,1)$, this family of twisted K-energy is indeed coercive in terms of geodesic distance (cf. Theroem 3.4 for more precise statement). While we wish this path were new, as usual, we find "footsteps of others", in particular the works of Fine, Stoppa and Lejmi-Székelyhidi [56, 77, 64]. The openness theorem is inspired by the renown work of LeBrun-Simanca [62] on deformation of extremal Kähler metrics.

1.1 A brief account on Kähler-Einstein problems

In 1976, S.-T. Yau solved the famous Calabi conjecture (by showing the existence of Ricci flat Kähler metric) if $C_1 = 0$. Around the same time, S.-T. Yau and T. Aubin independently solved the existence of the Kähler-Einstein metric if $C_1 < 0$. In 1990, S.-T. Yau first suggested that the existence of Kähler-Einstein metric is related to certain notion of stability of the underlying polarization. In 1997, G. Tian introduced the so-called *K-stability* (via special degeneration) and showed that the existence of Kähler-Einstein metric necessarily implies the K-stability of the underlying polarization through special degeneration. In 2002, S. K. Donaldson reformulated it into a notion of algebraic K-stability. In 2012, with S. K. Donaldson and S. Sun, we proved

Theorem 1.1 (Chen-Donaldson-Sun[25, 26, 27]) K-stable Fano manifolds admit Kähler-Einstein metrics.

The converse part of the above theorem is due to work of G. Tian[84], S. K. Donaldson[47], Stoppa [76], and the most general form is due to R. Berman [5]. There is also a Ricci flow approach to attack the existence problem of the Kähler-Einstein metrics. The fundamental problem in the Kähler Ricci flow is the so-called *Hamilton-Tian conjecture*: for any sequence of time $t_i \to \infty$, the corresponding sequence of Kähler metrics converge in Gromove-Hausdorff sense to a Kähler Ricci soliton with at most codimension 4 singularities. In 2014, Chen and Wang give an affirmative answer to this conjecture.

Theorem 1.2 (Chen-Wang[33]) For any Fano manifold, the Kähler Ricci flow will always converge to a Kähler-Ricci soliton with at most codimension 4 singularities in the sense of Cheeger-Gromov. Moreover, the complex structure may jump in the limit.

While there are many following up problems related to the existence problem of Kähler-Einstein metrics, the central problem in Kähler geometry is to solve the more general conjecture on the existence of cscK metrics.

Conjecture 1.3 (Yau-Tian-Donaldson) The underlying polarized Kähler manifold $(M, [\omega])$ is K-stable if and only if there exists a constant scalar curvature Kähler metric in $[\omega]$.

1.2 A new continuity path

For any Kähler manifold $(M, [\omega])$, consider the space of Kähler potentials

$$\mathcal{H} = \{ \varphi : \omega_{\varphi} = \omega + \sqrt{-1}\partial \bar{\partial} \varphi > 0, \text{ on } M \}.$$

E. Calabi proposed to study the existence of extremal Kähler metric in 1982. This has been a central problem in Kähler geometry since its inception. A Kähler metric is called "extremal" if the complex gradient vector field of its scalar curvature function is holomorphic. A special case is the so-called cscK metric (constant scalar curvature Kähler metric) when this vector field vanishes. To attack the existence problem of cscK metrics, we propose to study the following continuous path.

For any positive, closed (1,1)-form χ , define a continuous path $t \in [0,1]$ as

$$t \cdot \left(R_{\varphi_t} - \frac{[C_1(M)] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} \right) = (1-t) \cdot \left(\operatorname{tr}_{\varphi_t} \chi - \frac{[\chi] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} \right). \tag{1.1}$$

and for simplicity denote

$$\underline{\chi} = \frac{[\chi] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} \quad \text{and} \quad \underline{R} = \frac{[C_1(M)] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}}.$$

where $[\omega]^{[k]} = \frac{[\omega]^k}{k!}$. Let $C_t = (1-t)\underline{\chi} - t\underline{R}$, then Equation (1.1) is rewritten as

$$(1-t)\operatorname{tr}_{\omega_{t}}\chi - \operatorname{t} R_{\omega_{t}} = C_{t}. \tag{1.2}$$

Definition 1.4 A Kähler metric is called twisted cscK metric if its scalar curvature satisfies Equation (1.2). We call it twisted extremal Kähler metric if the left hand side of Equation (1.2) gives rise to a holomorphic vector field.

When t = 1, this reduces to the equation for cscK metrics. Let I denote the set of time parameter $t \in [0,1]$ such that Equation (1.1) can be solved at time t. As usual, our goal is to first prove that I is not empty which usually means finding a starting point where we can solve this equation. Then, to prove I is open which is crucial for this program to be viable. The hard part is of course to prove I is

closed which involves hard a priori estimate.

For each $t \in [0, 1]$, we call a solution to Equation (1.1) as twisted cscK metric with $(1 - t)\chi$. When $t = \frac{1}{2}$, this is exactly the equation considered by Stoppa [77], which in turns was motivated from J. Fine's work [56]. The first theorem we proved here is

Theorem 1.5 (Openness) For any $\chi > 0$ and $t \in (0,1)$, if there exists one solution to Equation (1.1) for time $t \in (0,1)$, then there exists a small $\delta > 0$ such that for any $t' \in (t-\delta, t+\delta) \cap [0,1)$, there exists a solution to Equation (1.1) for time t'.

This is a crucial result which establishes the validity of the path to solve the cscK metric problem in a given Kähler class. At technical level, it should be compared with the well-known deformation theorem proved by LeBrun-Simanca [62] where they discussed the deformation of cscK metrics when the Kähler class and/or complex structure changes. With the *openness* of I at present, the next important matter is wether I is non-empty. In particular, we need a starting point.

When t = 0, this reduces to the well-known equation

$$\operatorname{tr}_{\varphi}\chi = \chi. \tag{1.3}$$

which is the Euler-Lagrange equation for the well-known J functional introduced in [19], defined by the following formula of its derivative:

$$\frac{\mathrm{d}J_{\chi}}{\mathrm{d}t} = \int_{M} \frac{\partial \varphi}{\partial t} (\mathrm{tr}_{\varphi}\chi - \underline{\chi}) \omega_{\varphi}^{[\mathrm{n}]}. \tag{1.4}$$

We will delay discussions of J functional to the next section and for now, we just remark that there are several known obstructions to the solution of Equation (1.3). For instance,

$$\chi \cdot [\omega] > [\chi].$$

The main point is, if our goal is to attack existence of cscK metrics, we only need to choose one χ for which I is non-empty. For a moment of thought, it is obvious that we can always solve Equation (1.4) at t=0 if we choose χ to be in the same class as $[\omega]$. Thus, we can always find a smooth positive (1,1)-form χ such that set I is non-empty. In light of the openness theorem 1.5, this leaves the next problem both interesting and important:

Question 1.6 For any $\chi > 0$, if there exists one solution to Equation (1.1) for time t = 0, then there exists a small $\delta > 0$ such that for any $t \in [0, \delta) \cap [0, 1)$, there exists a solution to Equation (1.1) for time t.

We can prove directly that the linearized operator is strictly elliptic or semi-elliptic for $t \in [0, 1]$, as we will explain later. The case at t = 0, 1 are subtle for different reason. At time t = 1, the equation is degenerated elliptic where the *kernal* is induced by the underlying holomorphic vector field. We encourage readers to [39] for the discussions of this problem. Likewise, we also defer the discussion of the t = 0 to a later paper. The difficulty cause at t = 0 is of different nature: while all strictly elliptic operators, the inverse operator lost two derivatives when compairing to t > 0. Thus, we need to address this problem more carefully. The immediate thought would be use either a version of Nash-Moser inverse function theorem or the method via adiabatic limit as in a beautiful paper [56]. However, there is another different but plausible approach and we would like describe it now. Heuristically, one can take a flow approach as in [28]. One note that at t = 0, the J flow converges to the solution $\operatorname{tr}_{\varphi} \chi = \operatorname{const.}$ In other

words, for a sufficient small $C^{4,\alpha}$ neighborhood of this solution, the J flow is stable (in other words, for any flow initiate inside this neighborhood will never leave this neighborhood). Intuitively then, the twisted Calabi flow for t small enough is stable in this neighborhood and the argument in [28] might be adopted to settle this problem.

Another interesting twist is to compare this path with Donaldson's Continuity Path on conical KE metrics (see discussions in Section 2). The corresponding problem there is to show the existence of conical Kähler-Einstein metrics when conical angle is very small. Note that in the case of conical KE path, the existence for small angle is established through difficult work by R. Berman [6] and JMR [60] etc. It will be interesting to see if the method in [6] can be extend over to our situation. Suppose that there is a sequence of smooth, positive (1,1) form χ_{ϵ} such that $\lim_{\epsilon \to 0} \chi_{\epsilon} = D$ and the convergence is smooth away from divisor D, then twisted conical cscK metric path 2.16 and the aformentioned path is naturally connected thorough this limiting process. It will be interesting to see if one can prove from the existence of equation for t small to the existence of conial KE with small angle, or vice versa. As a matter of fact, we may generalize a conjecture of Donaldson to include the case of twisted cscK metric:

Conjecture 1.7 Suppose $[D] = [\chi]$ and D is generic, then the existence of twisted cscK metric at $t \in [0,1]$ is equivalent to the existence of conical cscK metric at time $t \in [0,1]$.

Note that in canonical class, the necessary part is established in [25] and the sufficient part is open. If we drop the assumption that D is generic, then one expect the sufficient part to be false. It is interesting if we can re-construct Tian-Yau's theorem on complete Calabi-Yau metric on Kähler manifold outside a divisor or the existence of conical KE metric with small conical angle via twisted cscK metric approached suggested here. While technically it is still complicated, the conceptual picture is nonetheless very clear.

One surprising observation is that we can avoid this problem of "jump" or regularity at t=0 in Fano manifold by working on a slightly different path (2.12). Note that in Fano manifold, in any Kähler class, we can find a metric ω such that

$$Ric \ \omega > 0.$$

If we choose $\chi = Ric \omega > 0$, then

Corollary 1.8 (Openness) Suppose χ is a positive Ricci form in any Fano manifold. Then for any $t \in [0,1)$, if there exists one solution to Equation (1.1) for time $t \in [0,1)$, then there exists a small $\delta > 0$ such that for any $t' \in [t-\delta, t+\delta) \cap [0,1)$, there exists a solution to Equation (2.12) for time t'.

Let E be the well-known K-energy functional introduced by T. Mabuchi [66], whose derivative is given below:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_{M} \frac{\partial \varphi}{\partial t} (R_{\varphi} - \underline{R}) \omega_{\varphi}^{[n]}$$

we then call

$$E_{\chi,t} = (1-t)J_{\chi} + tE$$

the twisted K-energy functional.

Following Stoppa [77] and Donaldson [47], we have

Proposition 1.9 For any χ and $t \in [0,1]$, $\omega \to (1-t) \operatorname{tr}_{\varphi_t} \chi - \operatorname{tR}_{\varphi_t} - \operatorname{C}_t$ can be viewed as a moment map associated with the infinite dimensional group of exact symplectic diffeomorphisms.

¹Suppose that $[D] = [\chi] = C_1(M)$, then the path we consider can be reduced to the usual Donaldson's path on conical KE metrics

²The path (2.12) at [0,1] is equivalent to the path (1.1) at $[\frac{1}{2},1]$.

Following Donaldson [47], Stoppa [77] and in particular, Lejmi-Székelyhidi [64], we can also formulate a notion of twisted K-stability which we will delay the discussions to Section 5. In [18], we proved that J functional is strictly convex along $C^{1,1}$ geodesic segments. Following the recent work of Berman-Berndtsson[7], Chen-Li-Paun [16], we arrive at the following expected statement

Proposition 1.10 For any $t \in [0,1]$, the twisted functional $E_{\chi,t}$ is convex along any $C^{1,1}$ geodesic segment and in particular, strictly convex when t < 1.

When t = 0, this is due to [18]; when t = 1, this was first conjectured by the author and proved in recent works [7, 16]. The proposition above is essentially a combination of these two results (cf. [7, 16]). As corollary, we proved that

Corollary 1.11 For any $t \in [0,1]$, the functional $E_{\chi,t}$ is bounded from below if Equation (1.1) has a solution for t.

Corollary 1.12 For any $t \in [0,1)$, the twisted cscK metric is unique in its Kähler class.

We can certainly reformulate the well-known YTD conjecture for the twisted cscK metric.

Conjecture 1.13 There exists a twisted cscK metric if and only if it is twisted K-stable.

We also believe that we should have a corresponding notion of twisted Paul's stability which is equivalent to the existence of twisted cscK metric. It is well known that the existence of Kähler-Einstein metrics implies also Paul's stability if Aut(M,J) is discrete. Conversely, Paul's stability implies that the K-energy is proper in all finite Bergman spaces [68] . For semi-stable orbits, we believe the following conjecture holds

Conjecture 1.14 If $(M, [\omega], J)$ is destabilized by $(M, [\omega], J')$ (where the second one admits a cscK metric). Then for any $\chi \in [\omega]$, there exists a unique twisted cscK metric for $(1-s)R_{\varphi} - \operatorname{str}_{\varphi}\chi = \operatorname{c}_{s}$.

Theorem 1.15 ([39]) Let $\chi > 0$ be any closed, positive (1,1)-form in $[\omega]$. If there exists a cscK metric, then for $t \in [0,1]$ close enough to 1, there exists a solution to Equation (1.1).

We really follow the idea of Bando-Mabuchi to solve the path for 1-t small enough. In their case, they need to solve the Aubin continuity path along the way from t=1 to t=0. In the present situation, it is technically not feasible to do "weak compactness" for a family of twisted cscK metrics yet. Nonethelese, the problem of weak compactness of twisted cscK metric is one of the fundamental problems one ultimately need to address (cf. [83] [38] and references therein for discussion on this important topics).

1.3 Related problems

Following our discussions about this one parameter path (Equation (1.1)), it makes sense to define

Definition 1.16 For any $\chi > 0$, define $R(\chi)$ to be the supremum of time $t \in [0, 1]$ for which this equation is solvable.

In view of the openness theorem (Theorem 1.8), we make the following conjecture:

Conjecture 1.17 For any two closed, positive (1,1)-forms χ_1,χ_2 in the same cohomology class, we have

$$R(\chi_1) = R(\chi_2).$$

In the classical Aubin's continuous path, this conjecture was proved by G. Székelyhidi [79]. More interestingly, together with T. Collins in [41] they also proved that if this path is solvable for $\chi_1 > 0$ at time t = 0, then it is also solvable for $\chi_2 > 0$ at t = 0 provided that $[\chi_1] = [\chi_2]$. It will be really interesting to answer

Question 1.18 Can we characterize the polarized Kähler manifold $(M, [\omega])$ where there exists a (1,1)form $\chi > 0$ such that $R(\chi) = 1$ but with no cscK metrics in $(M, [\omega])$?

Another important problem is to study the following twisted Calabi flow for any $t \in [0, 1]$, where the family of Kähler potentials $\varphi = \varphi_s$ evolves by:

$$\frac{\partial \varphi}{\partial s} = tR_{\varphi} - (1 - t) \operatorname{tr}_{\varphi} \chi + C_{t}. \tag{1.5}$$

It is interesting and important to establish the short time existence for proper initial metrics. The author believes this is true, along with a stability theorem at the neighborhood of twisted cscK metric (as in [28] for the usual Calabi flow). Note that in Riemann surface [61], J. Pook proved short and long time existence of the twisted Calabi flow where the form depends on time t > 0. More importantly, one can adopt two important conjectures about the Calabi flow to the twisted case here.

Conjecture 1.19 (Chen) The twisted Calabi flow exists globally for any smooth initial Kähler potential.

Similarly, we can re-formulate Donaldson's conjectural pictures at the case of Calabi flow to our settings:

Conjecture 1.20 Suppose the twisted Calabi flows have global existence. Then the asymptotic behavior of the twisted Calabi flow starting from (M, ω, χ, J) falls into the following possibilities:

- 1. The flow converges to a twisted cscK metric on the same complex manifold (M, J);
- 2. The flow converges, up to a difference of the attention to a twisted extremal Kähler metric;
- 3. The manifold does not admit a twisted cscK metric or twisted extremal metric but the transformed flow $(M, [\omega_t], \chi_t, J_t)$ converges to $(Y, [\omega_\infty], \chi', J')$ which forms a twisted extremal Kähler metric, possibly with at least codimension 2 singularities.

More interestingly, it will be nice to understand the important result of J. Streets [78] as well as more recent exciting result on the Calabi flow [65] in the twisted Calabi flow case. We believe the result can be made more precise because of the strict convexity of the twisted K-energy functional.

Another important conjecture we want to raise is

Conjecture 1.21 For any $t \in (0,1)$, the twisted functional $E_{\chi,t}$ is proper in terms of geodesic distance function if and only if Equation (1.1) has a solution.

Note that when t = 1, this is the conjecture on the properness of the K-energy functional (see Conjecture 3.3 below). We remark that this is another reason why we believe Conjecture 1.17 is correct. In Section 3, we will prove the necessary part of this conjecture.

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2 Discussions about various known paths

According to E. Calabi, the famous Kähler-Einstein problem can be reduced to a problem of complex Monge-Ampère equation:

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_{\alpha} \partial \bar{z}_{\beta}}\right) = e^{-\varphi + h_{\omega}} \det(g_{\alpha\bar{\beta}}) \tag{2.6}$$

where

$$\omega_g + \sqrt{-1}\partial\bar{\partial}\varphi > 0$$
 on M.

To attack the existence problem, one adopts a continuous path

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_{\alpha} \partial \bar{z}_{\beta}}\right) = e^{-t\varphi + h_{\omega}} \det(g_{\alpha\bar{\beta}}), \qquad t \in [0, 1]. \tag{2.7}$$

When t = 0, this is Calabi's volume conjecture. The strategy is to prove the set

$$\{t \in [0,1] \mid \text{Equation } (2.7) \text{ can be solved at time } t\}$$

is both open and closed. The openness is more or less standard. To derive closedness, we need to obtain a priori estimate which is independent of t. The memorial feature of the resolution of the Calabi conjecture by S.-T. Yau is to reduce the second order estimate to a C^0 estimate on Kähler potential, which is followed by a C^3 estimate of E. Calabi. This is of course well-known to Kähler geometers. However, the following might be less well-known. A cscK metric equation can be decomposed as a coupled second order equations which we will describe below.

$$\log \frac{\det(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_{\alpha} \partial \bar{z}_{\beta}})}{\det(g_{\alpha\bar{\beta}})} = F, \tag{2.8}$$

$$\Delta_{\varphi}F = -\underline{R} + \operatorname{tr}_{\varphi} \operatorname{Ric}_{g} \tag{2.9}$$

The following proposition is known:

Proposition 2.1 If g_{φ} is a cscK metric and it is quasi isometric to the background metric g, then the $C^{4,\alpha}$ norm on φ is uniformly controlled.

Here is a few words about the proof of this proposition: since g_{φ} is quasi-isometric, then Equation (2.9) is uniformly elliptic with a bounded right hand side. Therefore, by De Giorgi, $[F]_{C^{\alpha}(M,g)}$ is uniformly bounded. Substituting this into Equation (2.8), it becomes a complex Monge-Ampère equation with C^{α} bound on the right hand side. According to theory of Caffarelli, Evans-Krylov and the recent observation of Y. Wang [91] (cf. also Chen-Wang[37]), we know $[\varphi]_{C^{2,\alpha}(M,g)}$ is uniformly controlled. Iterating from Equation (2.8) and (2.9) once more, we get $[\varphi]_{C^{4,\alpha}(M,g)}$ is uniformly controlled.

Inspired by this proposition, we propose the following naively looking equation:

$$\log \frac{\det(g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z_{\alpha} \partial \bar{z}_{\beta}})}{\det(g_{\alpha\bar{\beta}})} = F, \tag{2.10}$$

$$\Delta_{\omega}F = t(-R + \operatorname{tr}_{\omega}\operatorname{Ric}\,\mathbf{g}) \tag{2.11}$$

for any $t \in [0, 1]$. Set I to be all $t \in [0, 1]$ such that the above equation can be solved. Then, obviously $0 \in I$. since it reduces to the well-known Calabi's volume conjecture albeit it is trivial in this setting.

The fundamental problem is if the set I necessarily open? It is hard to tell directly the answer to this crucial question. More importantly, it is quite difficult to imagine the role of the sign of "Ric g" played in this family of equations. Purely from PDE consideration, I think both positivity and negativity of "Ric g" have their own advantages and pitfalls. However, these two signs are distinctly different from geometric perspective. In fact, the *openness* depends crucially on the sign of "Ric g". It is open if it is positive. This can be readily seen if we re-write this path of pair of equations as

$$R_{\varphi} = (1 - t) \operatorname{tr}_{\varphi} \operatorname{Ric} g + t \underline{R}$$
 (2.12)

This is very similar to Equation (1.1) with $\chi = \text{Ric } g$. Thus, in light of Theorem 1.8, the *openness* holds as long as Ric g > 0. It is not clear what happens if Ric g < 0.

On the other hand, we observed that there is an untraced version of this continuous path, which is the well-known Aubin path. Note that in Kähler-Einstein settings, the Aubin path reads as

$$(1-t)\chi - t \operatorname{Ric} \omega = \frac{C_t}{n}\omega \tag{2.13}$$

This casts the "traditional Aubin path (2.7)" into new light: that this is also a continuous family of twisted cscK metrics over parameter $t \in [0,1]$. It is not difficult to see

Proposition 2.2 Along the Aubin path, if $(M, [\omega])$ is K semi-stable, then for every $t \in [0, 1)$, it is twisted K-stable.

Following strategy of [25, 26, 27], we may ask

Question 2.3 If there exists a sequence of twisted cscK metric ω_i for $(M, [\omega], \chi, t_i)$ where $t_i \to \bar{t}$ such that

- 1. $(M, [\omega], \chi, \bar{t})$ is twisted K-stable;
- 2. Partial C^0 estimates holds for (M, ω_i, χ, t_i) ,

then does $(M, [\omega], \chi, \bar{t})$ admit twisted cscK metric?

According to G. Szekelyhid [80], who in turns follows [26], the partial C^0 estimate holds for a sequence of Kähler metrics over *Aubin path*. If the answer to aformentioned question is true, then we can prove Yau's stability conjecture by following Aubin's path. This leaves a very interesting question

Question 2.4 For any sequence of twisted cscK metric ω_i for $(M, [\omega], \chi_i, t_i)$, when does the partial C^0 estimate hold?

In the singular case where $\chi = 2\pi[D]$ for a divisor $D \subset M$, this equation reduces to

$$\operatorname{Ric} \omega = -\frac{C_t}{nt}\omega + 2\pi \frac{1-t}{t}[D] \tag{2.14}$$

it provides a variant of *Donaldson's continuity path* of conical Kähler-Einstein metrics [52, Equation (27)]. The question is wether the parallel theory holds in this case?

This continuity path could be viewed as a natural generalization of the classical Aubin's continuity path

$$Ric \ \omega_t = t\omega + (1-t)\chi \tag{2.15}$$

concerning the study of Kähler-Einstein metrics. Taking trace of the equation (2.14) gives the notion of "conical cscK metric":

$$R_{\varphi} = -\frac{C_t}{t} + 2\pi \frac{1-t}{t} \operatorname{tr}_{\varphi}[D]$$
 (2.16)

We call ω this conical constant scalar curvature Kähler metric if it satisfies:

- 1. It is quasi isometric to a Kähler metric with cone angle $2\pi t$ near the divisor D;
- 2. The metric tensor is in $C^{\alpha,t}$ near divisor D for some $\alpha \in (0, \min(1, \frac{1}{t} 1))$;
- 3. The scalar curvature is constant outside the divisor D.

We could ask the following

Question 2.5 If there is no tangential holomorphic vector in (M, D), can we deform the conical cscK metric with angle t > 0? More interestingly, do we always have a conical cscK metric for t small enough?

3 Convexity and moment map pictures

3.1 K-energy functional and moment map picture

One memorial feature is that, via the work of S. Donaldson [46] and Fujiki, the scalar curvature (regarded as the Lie algebra element) of a Kähler metric can be viewed as a moment map for an action of the group of exact sympletic diffeomorphisms on the space of all compatible almost complex structures. In 1998, the author [17] established the existence of $C^{1,1}$ geodesic segment between two smooth Kähler potentials and proved that the K-energy functional is convex along $C^{1,1}$ geodesic segment if $C_1 \leq 0$. In general, the best regularity for geodesic segment might be $C^{1,1}$ only. At the time, the best we can prove is the following

Theorem 3.1 ([21, 30]) Suppose φ_0, φ_1 are two Kähler potential and $\varphi_t(t \in [0, 1])$ is the $C^{1,1}$ geodesic segment connecting φ_0 to φ_1 . Then,

$$(dE, \varphi'_t|_{t=0})|_{\varphi_0} \le (dE, \varphi'_t|_{t=1})|_{\varphi_1}$$

However, this theorem leads to the following conjecture [19] by the author.

Conjecture 3.2 The K-energy functional is convex on any $C^{1,1}$ geodesic segment.

New understanding in Kähler geometry leads to a proof to this full conjecture by the work of Berman-Berndtsson[7] and independently by Chen-Li-Paun [16]. Another related conjecture is the following

Conjecture 3.3 There exists a cscK metric if and only if the K-energy functional is coercive in terms of geodesic distance to the maximal invariant, totally geodesic subspace induced by automorphism group.

This should be seen as a continuation of Conjecture 1.21. In Kähler-Einstein manifold without holomophic vector field, Ding-Tian proved that the K-energy functional is proper in terms of Aubin functional. The corresponding conjecture (in terms of geodesic distance) has very little progress. Very recenty, in Fano manifold with no holomorphic vector field, T. Darvas [42] proved that the Ding functional is proper in terms of L^1 geodesic distance. This work gives a very good indication that the full conjecture above should be true (module some mild modifications). In toric surface and for t = 1, X.-H Zhu and B. Zhou can establish a weak existence of extremal Kähler metric if the modified K-energy functional is proper [95]. This conjecture more likely can be established in toric surface settings first.

Now we give a proof of the necessary part of Conjecture 1.21 with the assumption that J_{χ} is bounded from below.

Proof The following functional is well-known

$$J(\varphi) = \int_{M} \varphi\left(\omega^{n} - \omega_{\varphi}^{n}\right) = \int_{M} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \left(\sum_{k=0}^{n-1} \omega^{k} \wedge \omega_{\varphi}^{n-k-1}\right) > 0.$$

According to T. Davas's recent work [42, 43], this functional is equivalent to geodesic distance in \mathcal{H} with respect to Mabuchi's metric. Recall a decomposition formula of K-energy in [19]: For any $\phi \in \mathcal{H}_{\omega}$, we have

$$E(\varphi) = \int_{M} \ln \frac{\omega_{\varphi}^{n}}{\omega_{0}^{n}} \omega^{n} - J_{Ric \, \omega_{0}}(\varphi).$$

where $J_{Ric \,\omega_0}$ is the J_{χ} functional defined by Formula 1.4 for $\chi = Ric \,\omega_0$.

The first term of this decomposition formula (of K-energy) is called Entropy functional and we will denote it as $E_0(\varphi)$. We claim that Entropy functional is proper in terms of geodesic distance. According to Davas's recent work [42, 43], we only need to prove the entropy functional dominates this J functional. This is a well-known fact and we include a short proof here for completeness (cf. [21] for this calculation). According G. Tian, there is a positive constant $\alpha > 0$ which depends only on the Kähler class $[\omega]$ such that for any $\varphi \in \mathcal{H}$, we have

$$\int_{M} e^{-\alpha(\varphi - \sup \varphi)} \omega^{n} \le C,$$

or

$$\int_{M} e^{-\alpha(\varphi - \sup \varphi) - \log \frac{\omega_{\varphi}^{n}}{\omega^{n}}} \omega_{\varphi}^{n} \leq C.$$

If we set

$$\int_{M} \omega^{n} = 1,$$

we have

$$\int_{M} -\alpha(\varphi - \sup \varphi) - \log \frac{\omega_{\varphi}^{n}}{\omega^{n}} \omega_{\varphi}^{n} \le C.$$

Therefore,

$$\alpha \sup \varphi \leq \alpha \int_{M} \varphi \omega_{\varphi}^{n} + \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega^{n}} \omega_{\varphi}^{n}$$
$$\leq \alpha \int_{M} \varphi \omega_{\varphi}^{n} + \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega^{n}} \omega_{\varphi}^{n}.$$

In other words,

$$\alpha \int_{M} \varphi \omega^{n} \leq \alpha \sup \varphi \leq \alpha \int_{M} \varphi \omega_{\varphi}^{n} + \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega^{n}} \omega_{\varphi}^{n}.$$

Consequently,

$$\alpha J(\varphi) = \alpha \int_{M} \varphi(\omega^{n} - \omega_{\varphi}^{n}) \le \int_{M} \log \frac{\omega_{\varphi}^{n}}{\omega^{n}} \omega_{\varphi}^{n}.$$

Thus, the entropy functional is proper in terms of geodesic distance.

Now, we are ready to prove the necessary part of this theorem. We assume that

$$tr_{\varphi}\chi = \frac{[\chi] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}}$$

can be solved in $[\omega]$. By convexity, we know that J_{χ} has lower bound in \mathcal{H} first. Then, for $\epsilon > 0$ small enough, we have

$$J_{\chi} \geq \pm \epsilon J_{Ric \omega} - C.$$

This is because we can solve the corresponding equation

$$\operatorname{tr}_{\varphi}(\chi \pm \epsilon \operatorname{Ric} \omega) = \frac{[\chi] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}} + \epsilon \frac{[C_1(M)] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}}$$

via perturbation for small enough $\epsilon > 0$. The desired inequality then follows from convexity of $J_{\chi \pm \epsilon Ric \ \omega}$ functional over $C^{1,1}$ geodesic segment in \mathcal{H} .

Suppose ω_{φ} is a twisted cscK metric for $\chi > 0$ and $t_0 \in (0,1)$, then there exists a small $\delta > 0$ such that the twisted cscK metric equation can be solved for any $t' \in (t_0, t_0 + 2\delta) \subset (0,1)$. Fix $t' = t_0 + \delta$. Then,

$$E + \frac{1 - t'}{t'} J_{\chi} \ge -C. \tag{3.17}$$

Note that the coefficient $\frac{1-t'}{t'}$ is strictly less than $\frac{1-t_0}{t_0}$. In other words, this inequality holds for any number close enough to $\frac{1-t_0}{t_0}$. Now

$$\begin{array}{lcl} t_0 E + (1 - t_0) J_{\chi} & = & (t_0 - \epsilon) E + (1 - t_0 - \delta') J_{\chi} + \epsilon E + \delta' J_{\chi} \\ & = & (t_0 - \epsilon) (E + \frac{1 - t_0 - \delta'}{t_0 - \epsilon} J_{\chi}) + \epsilon E_0 + (\delta' J_{\chi} + \epsilon J_{Ric \, \omega}) + \epsilon \underline{R} I(\varphi). \end{array}$$

Now, for fixed δ' small enough, we can find ϵ small enough so that

$$\delta' J_{\gamma} + \epsilon J_{Ric \, \omega} \geq -C$$

and

$$E + \frac{1 - t_0 - \delta'}{t_0 - \epsilon} J_{\chi} \ge -C.$$

Thus, we prove

$$t_0E + (1 - t_0)J_{\gamma} \ge \epsilon E_0 - C$$

It follows that the twisted K-energy is proper in terms of geodesic distance.

In summary, we prove

Theorem 3.4 Suppose that

$$tr_{\varphi}\chi = \frac{[\chi] \cdot [\omega]^{[n-1]}}{[\omega]^{[n]}}$$

can be solved in $[\omega]$. For any $t \in (0,1)$, the twisted K-energy functional is coercive in terms of geodesic distance if one of the following condition holds:

- 1. There exists a constant scalar curvature metric;
- 2. The K-energy functional is bounded from below;
- 3. There existed a twisted cscK metric for $t \in (0,1)$.

Conjecture 3.5 For any $C^{1,1}$ Kähler potential, we can have a Calabi flow or twisted Calabi flow initiated from this potential and the flow becomes instantly smooth. Consequently, the $C^{1,1}$ minimizer of the twisted K-energy is always smooth.

In canonical Kähler class, both statements has been asserted true by the efforts of many mathematicians. The weak Kähler Ricci flow was initially introduced to obtain partial regularity for $C^{1,1}$ minimizer on the K-energy functional in [20] and subsequently [23] and [24]. Now, the weak Ricci flow is itself an intensive subject of study: there are lots of interesting results on the regularity properties starting from weak data, see [53, 54, 59, 45, 67] for a partial list and reference therein.

3.2 J functional

In [18], the author introduce the so-called *J-flow* as

$$\frac{\partial \varphi}{\partial t} = \underline{\chi} - \operatorname{tr}_{\varphi} \chi.$$

This is used to study the lower bound of the K-energy functional on ample Kähler manifold. A striking feature of this J flow is its convexity along any $C^{1,1}$ geodesic segment.

Proposition 3.6 ([18]) J is a strictly convex functional along any $C^{1,1}$ geodesic. In particular, J has at most one critical point in \mathcal{H} .

For the convenience of readers, we re-produce the proof here.

Proof Suppose φ_t is a $C^{1.1}$ geodesic. In other words, φ_t is a weak limit of the following continuous equation as $\epsilon \to 0$ (with uniform bounds on the second mixed derivatives of Kähler potentials):

$$\left(\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{2} \mid \nabla \frac{\partial\varphi}{\partial t}\mid_{\varphi}^2\right) \frac{\omega_{\varphi_t}^n}{n!} = \epsilon \cdot \frac{\omega_0^n}{n!}.$$

Denote g_t as the corresponding Kähler metric corresponds to the Kähler form ω_{φ_t} . Again, we drop the dependence of t from g_t for convenience from now on. Recall the definition of J, we have

$$\frac{\mathrm{d}J_{\chi}}{dt} = \int_{V} \frac{\partial \varphi}{\partial t} \left(g^{\alpha \overline{\beta}} \chi_{\alpha \overline{\beta}} \right) \frac{\omega_{\varphi}^{n}}{n!}.$$

Then (denote $\sigma = g^{\alpha \overline{\beta}} \chi_{\alpha \overline{\beta}}$ in the following calculation):

$$\begin{array}{ll} \frac{d^2J}{dt^2} & = & \int_V \left(\frac{\partial^2\varphi}{\partial t^2} \sigma - \frac{\partial\varphi}{\partial t} g^{\alpha\overline{\beta}} (\frac{\partial\varphi}{\partial t})_{,\overline{\beta}r} g^{r\overline{\delta}} \chi_{\alpha\overline{\delta}} + \frac{\partial\varphi}{\partial t} \, \sigma \Delta_g \, \frac{\partial\varphi}{\partial t} \right) \, \frac{\omega_\varphi^n}{n!} \\ & = & \int_V \left(\frac{\partial^2\varphi}{\partial t^2} \sigma - \frac{\partial\varphi}{\partial t} g^{\alpha\overline{\beta}} (\frac{\partial\varphi}{\partial t})_{,\overline{\beta}r} g^{r\overline{\delta}} \chi_{\alpha\overline{\delta}} \\ & & - (\frac{\partial\varphi}{\partial t})_{,r} \sigma g^{r\overline{\delta}} (\frac{\partial\varphi}{\partial t})_{,\overline{\delta}} - \frac{\partial\varphi}{\partial t} g^{\alpha\overline{\beta}} \chi_{\alpha\overline{\beta},\overline{\delta}} g^{r\overline{\delta}} (\frac{\partial\varphi}{\partial t})_{,r} \right) \, \frac{\omega_\varphi^n}{n!} \\ & = & \int_V \left((\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial\varphi}{\partial t}|_g^2) \sigma - \frac{\partial\varphi}{\partial t} g^{\alpha\overline{\beta}} (\frac{\partial\varphi}{\partial t})_{,\overline{\beta}r} g^{r\overline{\delta}} \chi_{\alpha\overline{\delta}} \right. \\ & & - \frac{\partial\varphi}{\partial t} \left(g^{\alpha\overline{\beta}} \chi_{\alpha\overline{\delta}} g^{r\overline{\delta}} \right)_{,\overline{\beta}} (\frac{\partial\varphi}{\partial t})_{,r} \right) \, \frac{\omega_\varphi^n}{n!} \\ & = & \int_V \left((\frac{\partial^2\varphi}{\partial t^2} - \frac{1}{2} |\nabla \frac{\partial\varphi}{\partial t}|_g^2) \, (g^{\alpha\overline{\beta}} \chi_{\alpha\overline{\beta}}) + (\frac{\partial\varphi}{\partial t})_{,\overline{\beta}} \left(g^{\alpha\overline{\beta}} \chi_{\alpha\overline{\delta}} g^{r\overline{\delta}} \right) (\frac{\partial\varphi}{\partial t})_{,r} \right) \, \frac{\omega_\varphi^n}{n!} \\ & \geq & \int_V \left(\frac{\partial\varphi}{\partial t})_{,\overline{\beta}} \left(g^{\alpha\overline{\beta}} \chi_{\alpha\overline{\delta}} g^{r\overline{\delta}} \right) (\frac{\partial\varphi}{\partial t})_{,r} \, \frac{\omega_\varphi^n}{n!} \geq 0. \end{array}$$

The last equality holds along any $C^{1,1}$ geodesic.

Since its inception, the flow is now well studied. According to Song-Weinkove [75], the necessary and sufficient condition for the flow to converge is that there exists a form $\omega' \in [\omega]$ such that

$$(n\chi\omega' - (n-1)\chi) \wedge \omega'^{n-2} > 0.$$

For more updated work on this subject, we refer readers to Weinkove [92], Song-Weinkove [75] and Fang-Lai-Song-Weinkove [55] for further readings.

Following this proposition and the recent works [7, 16], we can easily prove that the twisted K-energy functional is convex (Proposition 1.10), and bounded from below if there is a twisted cscK metric (Corollary 1.11) and finally can prove the uniqueness of twisted cscK metric for t < 1.

4 Deformation

In this section, we will prove the openness for deformation of twisted cscK metrics (Theorem 1.8). Now we assume $t \in (0,1)$. Set

$$\mathcal{H}^{4,\alpha}(M) = \{ \varphi \in C^{4,\alpha}(M,\mathbb{R}) \mid \omega_{\varphi} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \}$$

For any closed positive (1,1)-form χ on M, we define a map

$$F: \mathcal{H}^{4,\alpha}(M) \times [0,1] \longrightarrow C^{\alpha}(M,\mathbb{R}) \times [0,1]$$
$$(\varphi,t) \longmapsto (t(R_{\varphi} - \underline{R}) - (1-t)(\operatorname{tr}_{\varphi}\chi - \chi), t)$$

where R_{φ} is the scalar curvature of ω_{φ} . The openness theorem 0 < t < 1 is equivalent to the following ³.

Theorem 4.1 If $F(\varphi_0, t_0) = (0, t_0)$ for some $t_0 \in (0, 1)$, then for $t \in [0, 1)$ which is sufficiently close to t_0 , we can find $\varphi = \varphi(t)$ such that $F(\varphi, t) = (0, t)$.

Consider the linearization of F:

$$\mathcal{D}F|_{(\varphi,t)}: C^{4,\alpha}(M) \times \mathbb{R} \longrightarrow C^{\alpha}(M) \times \mathbb{R}$$
$$(u,s) \longmapsto (\mathcal{L}_{(\varphi,t)}u + s(R_{\varphi} - \underline{R} + \operatorname{tr}_{\varphi}\chi - \chi), s)$$

where $\mathcal{L}_{(\varphi,t)}$ is the linearized operator of the twisted scalar curvature function, i.e.

$$\mathcal{L}_{(\varphi,t)}u = -t\Delta_{\varphi}^{2}u - \langle \sqrt{-1}\partial\bar{\partial}u, t\operatorname{Ric}_{\varphi} - (1-t)\chi \rangle_{\varphi}$$

Set $T = \mathcal{D}F|_{(\varphi_0,t_0)}$, then

Lemma 4.2 $\mathcal{R}(T)$, which is the range of T, is closed.

Before we prove this lemma, it is important to note that

Lemma 4.3 The kernel of T is one dimensional for any (φ, t) which is sufficiently close to (φ_0, t_0) in $C^{4,\alpha}(M) \times \mathbb{R}$.

Proof For any $(u, s) \in \text{Ker } T$, without loss of generality we might assume that

$$\int_{M} u\omega_{\varphi}^{n} = 0.$$

By definition, we have s = 0 and

$$\mathcal{L}_{(\varphi,t)}u=0.$$

Thus,

$$\int u \mathcal{L}_{(\varphi,t)} u \omega_{\varphi}^{n} = \int \left(-t \Delta_{\varphi}^{2} u - t u_{,\bar{\alpha}\beta} (\operatorname{Ric}_{\varphi})_{\alpha\bar{\beta}} + (1-t) u_{,\bar{\alpha}\beta} \chi_{\alpha\bar{\beta}} \right) u \omega_{\varphi}^{n}$$

$$= \int \left(-t |u_{,\bar{\alpha}\bar{\beta}}|_{\varphi}^{2} - (1-t) u_{,\bar{\alpha}} u_{,\beta} \chi_{\alpha\bar{\beta}} + u_{,\bar{\delta}} (t R_{\varphi} - (1-t) \operatorname{tr}_{\varphi} \chi)_{,\delta} u \right) \omega_{\varphi}^{n}.$$

 $^{^3}$ We will deal with the case t=0 elsewhere.

It follows that

$$(1-t)\int u_{,\bar{\alpha}}u_{,\beta}\chi_{\alpha\bar{\beta}}\omega_{\varphi}^{n} = -\int t|u_{,\bar{\alpha}\bar{\beta}}|_{\varphi}^{2}\omega_{\varphi^{n}} + \int u_{,\bar{\delta}}(tR_{\varphi} - (1-t)\operatorname{tr}_{\varphi}\chi)_{,\delta}u\omega_{\varphi}^{n}$$

$$\leq -\int t|u_{,\bar{\alpha}\bar{\beta}}|_{\varphi}^{2} + \epsilon(\int u^{2} + \int |\nabla u|_{\varphi}^{2})$$

$$\leq -(1-\epsilon)t\int |u_{,\bar{\alpha}\bar{\beta}}|_{\varphi}^{2} + \epsilon(C+1)\int |\nabla u|_{\varphi}^{2}.$$

Here C is the Poincaré constant with respect to the metric ω_{φ} and ϵ is controlled by the quantity:

$$\sup_{M} |\nabla (tR_{\varphi} - (1-t)\mathrm{tr}_{\varphi}\chi)|$$

which could be made as small as we want for choosing (φ, t) sufficiently close to (φ_0, t_0) . Thus, for any $t_0 < 1$, we can choose a small neighborhood of t_0 such that

$$(1-t)c > \epsilon(1+C)$$

here 2c is the lower bound of χ in terms of ω_{φ_0} . Consequently, we have

$$\int |\nabla u|_{\varphi}^2 = \int |u_{,\bar{\alpha}\bar{\beta}}|_{\varphi}^2 = 0$$

which implies u = 0.

Now we return to the proof of Lemma 4.2.

Proof of Lemma 4.2 Suppose $(f_i, s_i) \in \mathcal{R}(T)$ such that f_i converges strongly in $C^{\alpha}(M)$ to f and $s_i \to s$ as $i \to \infty$. We want to prove that $(f, s) \in \mathcal{R}(T)$. By definition, we may assume that $(f_i, s_i) = T(u_i, s_i)$ where $u_i \in C^{4,\alpha}(M)$. From the definition of the operator T, and since T(u + C, s) = T(u, s), we can set

$$\int u_i \omega_{\varphi_0} = 0$$

In other words,

$$\mathcal{L}_{(\varphi_0,t_0)}u_i + s_i(R_{\varphi_0} - \underline{R} + \operatorname{tr}_{\varphi_0}\chi - \chi) = f_i$$

For any function $u \in C^{4,\alpha}(M)$, we have

$$\int u \mathcal{L}_{(\varphi_0, t_0)} u \omega_{\varphi_0}^n = \int \left(-t_0 \Delta_{\varphi_0}^2 u - t_0 u_{,\bar{\alpha}\beta} (\operatorname{Ric}_{\varphi_0})_{\alpha\bar{\beta}} + (1 - t_0) u_{,\bar{\alpha}\beta} \chi_{\alpha\bar{\beta}} \right) u \omega_{\varphi_0}^n$$

$$= \int \left(-t_0 |u_{,\bar{\alpha}\bar{\beta}}|_{\varphi_0}^2 - (1 - t_0) u_{,\bar{\alpha}} u_{,\beta} \chi_{\alpha\bar{\beta}} + u_{,\bar{\delta}} (t_0 R_{\varphi_0} - (1 - t_0) \operatorname{tr}_{\varphi_0} \chi)_{,\delta} \mathbf{u} \right) \omega_{\varphi_0}^n$$

Since $F(\varphi_0, t_0) = (0, t_0)(t_0 < 1)$ and $\chi > 0$, we have

$$-\int u\mathcal{L}_{(\varphi_0,t_0)}u\omega_{\varphi_0}^n \geq \epsilon_0 \int |\nabla_{\varphi_0}u|_{\varphi_0}^2 \omega_{\varphi_0}^n.$$

Therefore, we have

$$\begin{split} \epsilon_0 \int |u_i|^2 \omega_{\varphi_0}^n &\leq C \cdot \epsilon_0 \int |\nabla_{\varphi_0} u_i|^2 \omega_{\varphi_0}^n \\ &\leq C \epsilon \int \left(s_i (R_{\varphi_0} - \underline{R} + \operatorname{tr}_{\varphi_0} \chi - \underline{\chi}) - f_i \right) u_i \omega_{\varphi_0}^n \\ &\leq \frac{1}{2} \epsilon_0 \int |u_i|^2 \omega_{\varphi_0}^n + C. \end{split}$$

Thus,

$$\int |u_i|^2 \omega_{\varphi_0}^n \le C.$$

It follows that

$$\int |\nabla_{\varphi_0} u_i|^2 \omega_{\varphi_0}^n \le C.$$

Let's first consider the case when $0 < t_0 < 1$. Given the various bounds above, it's not hard to prove that

$$\int |\Delta_{\varphi_0} u_i|^2 \omega_{\varphi_0}^n \le C$$

or

$$||u_i||_{W_{\varphi_0}^{2,2}}^2 < C.$$

Now we can re-write the equation for u_i as

$$t_0 \Delta_{\varphi_0}(\Delta_{\varphi_0} u_i) = -t_0 u_{i,\bar{\alpha}\beta} (\operatorname{Ric}_{\varphi_0})_{\alpha\bar{\beta}} + (1 - t_0) u_{i,\bar{\alpha}\beta} \chi_{\alpha\bar{\beta}} + s_i (R_{\varphi_0} - \underline{R} + \operatorname{tr}_{\varphi_0} \chi - \underline{\chi}) - f_i.$$

Note that the right hand side is uniformly bounded in L^2 space. Thus, we have

$$\|\Delta_{\varphi_0} u_i\|_{W_{\varphi_0}^{2,2}}^2 < C.$$

or

$$||u_i||_{W_{\varphi_0}^{4,2}}^2 < C.$$

Following the standard bootstrapping argument in *elliptic theory*, we have

$$||u_i||_{C^{4,\alpha}(M)} \le C(||u_i||_{L^2(M)} + ||f_i - s_i(R_{\varphi_0} - \underline{R} + \operatorname{tr}_{\varphi_0} \chi - \chi)||_{C^{\alpha}(M)}) \le C$$

It follows that for any $\alpha' < \alpha$, we can choose a subsequence $u_i \xrightarrow{C^{4,\alpha'}(M)} u$, as $i \to \infty$ and $u \in C^{4,\alpha}(M)$. Let (f',s) = T(u,s), then $f_i \xrightarrow{C^{\alpha'}(M)} f'$, as $i \to \infty$. Therefore, $(f,s) = (f',s) = T(u,s) \in \mathcal{R}(T)$. So we proved that $\mathcal{R}(T)$ is closed.

Following the standard theory on linear operators among Hilbert spaces, we have

Corollary 4.4 The following decomposition holds:

$$C^{\alpha}(M) \times \mathbb{R} = \mathcal{R}(T) \oplus \mathbb{R}.$$

Proof By definition of the operator T, we know that for any $(f,s) \in \mathcal{R}(T)$, we have $\int_M f \omega_{\varphi_0}^n = 0$. Set $C_0^{\alpha}(M) = \{f \in C^{\alpha}(M) : \int_M f \omega_{\varphi_0}^n = 0\}$. Then, $\mathcal{R}(T) \subset C_0^{\alpha}(M) \times \mathbb{R}$. The equality holds since T is a linear, self adjoint operator. So the dimension of *Kernel* is the same as the dimension of *coKernal*. Since dim Ker T = 1, thus dim coKer T = 1. It follows that

$$C^{\alpha}(M) \times \mathbb{R} = \mathcal{R}(T) \oplus \mathbb{R}.$$

Denote $\tilde{\mathcal{R}}(T) = \pi_{C^{\alpha}(M)} \mathcal{R}(T) = C_0^{\alpha}(M) \times \mathbb{R}$.

Proof of Theorem 4.1 Without loss of generality, we can assume $\int_M \varphi_0 \omega_{\varphi_0}^n = 0$. Consider $F^1(\varphi, t) = \pi_{C^{\alpha}(M)} F(\varphi, t)$ as the projection of F to the $C^{\alpha}(M)$ component. Thus

$$F^{1}(\varphi,t) = t(R_{\varphi} - \underline{R}) - (1-t)(\operatorname{tr}_{\varphi}\chi - \underline{\chi}).$$

Consider the map

$$\Psi: \mathcal{H}^{4,\alpha}(M) \times [0,1] \longrightarrow (C_0^{\alpha}(M) \oplus \mathbb{R}) \times [0,1]$$
$$(\varphi,t) \longmapsto (\tilde{\pi} \circ F^1(\varphi,t) + \int \varphi \omega_{\varphi_0}^n, t),$$

where $\tilde{\pi}$ is the projection to $C_0^{\alpha}(M)$, i.e. $\tilde{\pi}(f) = f - \oint_M f \omega_{\varphi_0}^n$ for any function $f \in C^{\alpha}(M)$. Note that

$$\oint_{M} F^{1}(\varphi, t_{0}) \omega_{\varphi_{0}}^{n} = 0$$

Thus, corresponding to the variation $\delta \varphi = u$, the variation of $\tilde{\pi} \circ F^1$ at $t = t_0$ is:

$$\delta\left(\tilde{\pi}\circ F^{1}\right)(\varphi,t)\mid_{t=t_{0}}=\mathcal{L}_{(\varphi_{0},t_{0})}u+s(R_{\varphi_{0}}-\underline{R}+\operatorname{tr}_{\varphi_{0}}\chi-\underline{\chi}).$$

It follows that

$$\mathcal{D}\Psi|_{(\varphi_0,t_0)}(u,s) = \left(\mathcal{D}F^1|_{\varphi_0,t_0}(u,s) + \int u\omega_{\varphi_0}^n,s\right)$$
$$= \left(\mathcal{L}_{(\varphi_0,t_0)}u + s(R_{\varphi_0} - \underline{R} + \operatorname{tr}_{\varphi_0}\chi - \underline{\chi}) + \int u\omega_{\varphi_0}^n,s\right).$$

By our discussions earlier, $\mathcal{D}\Psi|_{(\varphi_0,t_0)}: C^{4,\alpha}(M) \times \mathbb{R} \to (C_0^{\alpha}(M) \oplus \mathbb{R}) \times \mathbb{R}$ is bijective. Thus, by inverse function theorem, we can find its inverse

$$\Psi^{-1}: (C_0^{\alpha}(M) \oplus \mathbb{R}) \times [0,1] \to C^{4,\alpha}(M) \times [0,1]$$

near $(0, t_0)$. In other words, there exists a small open neighborhood

$$V_{0,t_0} \subset C^{\alpha}(M) \times [0,1] = (C_0^{\alpha}(M) \oplus \mathbb{R}) \times [0,1]$$

where Ψ^{-1} is well defined such that

$$\Psi^{-1}(V_{0,t_0}) = U_{(\varphi_0,t_0)} \subset C^{4,\alpha}(M) \times [0,1].$$

Denote

$$\tilde{F} = F \circ \Psi^{-1} : V_{(0,t_0)} \longrightarrow (C_0^{\alpha}(M) \oplus \mathbb{R}) \times [0,1]$$
$$(w+a,t) \longmapsto (w+\int_M F^1 \circ \Psi^{-1}(w+a,t) \,\omega_{\varphi_0}^n, t).$$

and

$$f(w, a, t) = \int_{M} F^{1} \circ \Psi^{-1}(w + a, t) \omega_{\varphi_{0}}^{n}.$$

Then the linearized operator is

$$\mathcal{D}\tilde{\mathcal{F}}|_{(w+a,t)}: (C_0^{\alpha}(M) \oplus \mathbb{R}) \times \mathbb{R} \longrightarrow (C_0^{\alpha}(M) \oplus \mathbb{R}) \times \mathbb{R}$$
$$(u+b,s) \longmapsto (u+(\frac{\partial f}{\partial w}(w,a,t)u + \frac{\partial f}{\partial a}(w,a,t)b + \frac{\partial f}{\partial t}(w,a,t)s), s).$$

Next we consider its kernel $\operatorname{Ker}(\mathcal{D}\tilde{\mathcal{F}}|_{(w+a,t)})$. Since $\tilde{F} = F \circ \Psi^{-1}$ and Ψ is bijection, we need to consider the dim $\operatorname{Ker}(\mathcal{D}\mathcal{F}|_{(\varphi,t)})$. Clearly, we have

$$\operatorname{Ker}(\mathcal{DF}|_{(\varphi,t)}) = \{(u,0) \in C^{4,\alpha}(M) \times \mathbb{R} | \mathcal{L}_{(\varphi,t)}u = 0\}.$$

Following the proof of the previous lemma, we can easily prove that for any $(u,0) \in \text{Ker}(\mathcal{DF}|_{(\varphi,t)})$, we have

$$u = \oint_M u\omega_{\varphi_0}^n.$$

On the other hand, it is clear that

$$\{(C,0)\in C^{4,\alpha}(M)\times\mathbb{R}\}\subset \mathrm{Ker}(\mathcal{DF}|_{(\varphi,t)}).$$

Therefore, dim $\operatorname{Ker}(\mathcal{DF}|_{(\varphi,t)}) = 1$. It follows that $\dim_{\mathbb{R}} \operatorname{Ker}(\mathcal{D\tilde{F}}|_{(w+a,t)}) = 1$ for (w+a,t) sufficiently close to $(0,t_0)$. We claim that $\frac{\partial f}{\partial a}(w,a,t) = 0$. Otherwise, $\dim_{\mathbb{R}} \operatorname{Ker}(\mathcal{D\tilde{F}}|_{(w+a,t)}) = 0$. Note for any $(u+b,s) \in \operatorname{Ker}(\mathcal{D\tilde{F}}|_{(w+a,t)})$, we have

$$(u + (\frac{\partial f}{\partial w}(w, a, t)u + \frac{\partial f}{\partial a}(w, a, t)b + \frac{\partial f}{\partial t}(w, a, t)s), s) = (0 + 0, 0).$$

It follows that u = s = 0 and

$$\frac{\partial f}{\partial a}(w, a, t)b = 0.$$

If $\frac{\partial f}{\partial a}(w, a, t) \neq 0$, then b = 0. It follows that $\text{Ker}(\mathcal{D}\tilde{\mathcal{F}})|_{(w+a,t)} = 0$. This is a contradiction so our claim holds. It follows that f(w, a, t) = f(w, t). Therefore,

$$\tilde{F}(w+a,t) = (w+f(w,t),t)$$

We want to find the preimage of \tilde{F} for (0,t). We claim that f(0,t)=0. First, we look at $(\varphi,t)=\Psi^{-1}(0,a,t)$. It means that

$$\Psi(\varphi, t) = (0, a, t).$$

In other words, we have

$$(\tilde{\pi} \circ F^1(\varphi, t) + \int_M \varphi \omega_{\varphi_0}^n, t) = (0, a, t).$$

It follows that $\int_M \varphi \omega_{\varphi_0}^n = a$ and $\tilde{\pi} \circ F^1(\varphi, t) = 0$. It follows that

$$F^{1}(\varphi,t) - \int_{M} F^{1}(\varphi,t)\omega_{\varphi_{0}}^{n} = 0.$$

which implies that $F^1(\varphi,t)(x) \equiv C$. Note that $\int_M F^1(\varphi,t)\omega_\varphi^n = 0$ by definition. Consequently, $F^1(\varphi,t)(x) \equiv 0$, and then

$$f(0,t) = \int_{M} F^{1}(\varphi,t)\omega_{\varphi_{0}}^{n} = 0.$$

So

$$\tilde{F}(0, a, t) = (0 + f(0, t), t) = (0, t).$$

We therefore have $F(\Psi^{-1}(0, a, t)) = (0, t)$. This completes the proof.

5 Twisted K-stability

5.1 Twisted K-stability

Corresponding to the twisted K-energy functional, for any test configuration λ of M (see [81] for a precise definition in the algebraic case when Ω represents the first Chern class of an holomorphic line bundle L, roughly speaking, it means that M is embedded in \mathbb{P}^{N_k} by the linear system |-kL| and

 $\lambda: \mathbb{C}^* \to GL(N_k, \mathbb{C})$ is a one parameter subgroup) we could define twisted Futaki invariant. Suppose $\lambda(s) = s^A$, define a function on \mathbb{P}^{N_k} ,

$$h_A = \frac{zAz^*}{zz^*}$$

then we make the following definition:

Definition 5.1 (twisted Futaki invariant)

$$Fut_t(\Omega, \chi; \lambda) = (1 - t)Fut(\chi; \lambda) + tFut(\Omega; \lambda)$$

where $Fut(\Omega; \lambda)$ is the algebraically defined Donaldson-Futaki invariant of λ , and

$$Fut(\chi;\lambda) = Ch(\chi;\lambda) - \underline{\chi}Ch(M;\lambda)$$

$$= \lim_{t \to 0} \{ \int_{M_s} h_A \lambda(s^{-1})^* \chi \wedge (\frac{1}{k}\omega_{FS}|_{M_s})^{[n-1]} - \underline{\chi} \int_{M_s} h_A (\frac{1}{k}\omega_{FS}|_{M_s})^{[n]} \}$$

$$= \lim_{t \to 0} \{ \int_{M} \chi \wedge \lambda(s)^* (h_A \omega_s^{[n-1]}) - \underline{\chi} \int_{M} \lambda(s)^* (h_A \omega_s^{[n]}) \}$$

Remark 5.2 In the special case when the central fiber $M_0 = \lim_{t\to 0} \lambda(s)M$ is a normal variety, $Fut_t(\Omega, \chi; \lambda)$ reduces to the usual log-Futaki invariant of the log pair $(M_0, (1-t)D_0)$ (first introduced by [52] for the study of conical Kähler-Einstein metrics) where $D_0 = \lim_{s\to 0} \lambda(s)D$ for a generic divisor D in the linear system $|L_{\chi}|$.

If χ and ω both belongs to $2\pi C_1(M)$, the twisted cscK metrics equation reduces to the Aubin's continuity path, for which the Futaki's invariant is already explicitly defined, see [81, Formula (4.10)].

Definition 5.3 (twisted K-semistable/stable) A triple (Ω, χ, s) as above is called twisted K-semistable if

$$Fut_t(\Omega, \chi; \lambda) \leq 0$$

for any test configuration λ ; and it is called twisted K-stable if

$$Fut_t(\Omega, \chi; \lambda) < 0$$

for any λ nontrivial.

It follows from the definition that $\operatorname{Fut}(\chi;\lambda)$ depends linearly on χ for any fixed λ . It is proved in [81, Thm 6] that

$$\operatorname{Fut}(\gamma; \lambda) < \operatorname{Fut}(D; \lambda)$$

for any divisor D in the linear system of L_{χ} , and for any fixed test configuration λ the equality holds for generic element D in this linear system. One consequence is that $\operatorname{Fut}(\chi;\lambda)$ is independent of the particular choice of χ , therefore the notion of twisted K-stability for a triple (Ω,χ,s) is independent of the choice of χ in a fixed cohomology class also .

Proposition 5.4 (Linear Interpolation) If (Ω, χ, t_0) and (Ω, χ, t_1) are both twisted K-semistable, and one of them is twisted K-stable, then (Ω, χ, t) is twisted K-stable for any $t \in (t_0, t_1)$.

Proof Since $\operatorname{Fut}_t(\Omega, \chi; \lambda)$ depends on t linearly for any fixed test configuration λ , the twisted K-stability for the two parameters t_0 and t_1 implies a preferred sign of Fut_t for any t in between.

Conjecture 5.5 If (Ω, χ, t) with t < 1 is twisted K-stable iff Equation (1.1) admits a solution for the parameter t.

It should be noticed that the twisted cscK equation (Equation (1.1) for $t = \frac{1}{2}$, the middle point of our continuous path) was already studied by [56, 77]. Stoppa in [77, Theorem 1.3] gave a cohomological obstruction, Kähler slope stability, to the existence extending the result of Ross-Thomas[72] for the untwisted case. In another direction, [64, Conjecture 1] conjectured a numerical criterion for the existence of solution to Equation (1.3)(Equation (1.1) for t = 0, the starting point of our continuous path), based on the study of deformation to the normal cone, a particular type of test configuration, extensively used in the work [72]. And recently this conjecture was proved in [41, Theorem 1.3] for toric manifolds, therefore our conjecture here holds on toric manifolds for the starting point.

Remark 5.6 While we were preparing this note, we noticed the recent work by R. Dervan [44] who introduced also the notion of (uniform) twisted K-stability and proved that the existence of twisted cscK metric implies the (uniform) twisted K-stability by using the lower bound of the twisted Calabi functional.

6 Twisting cscK metric with higher degree form

While a (1,1)-form $\chi > 0$ is convenient, from PDE point of view, there is no particular reasons to restrict oneselves to this setting only. For any integer $k \in \{1, \dots, n\}$, consider a closed (k, k)-form $\mu_k = \chi^k$, one can define a new functional

$$\frac{dJ_{\mu_k}}{dt} = \int_M \dot{\varphi}(\mu_k \wedge \omega_{\varphi}^{n-k} - c_k \omega_{\varphi}^n).$$

where the constant

$$c_k = \frac{[\mu_k] \cdot [\omega]^{n-k}}{[\omega]^n}$$

Remark that the *closedness* of μ_k guarantee that J_{μ_k} is well-defined. Note that this functional has been studied in the literature before (cf. [55]). The Euler-Lagrange equation is

$$\frac{\mu_k \wedge \omega_{\varphi}^{n-k}}{\omega_{\varphi}^n} = c_k$$

When k=0, this is the well-known I functional. When k=1, then this is just the usual J functional we discussed here. When k=n, then μ is a volume form, for which the Euler-Lagrange equation reduces to

$$\omega_{\varphi}^{n} = \frac{1}{c_{n}} \mu_{n}$$

which is a standard complex Monge-Ampère equation. In fact, this is precisely the Calabi's volume form conjecture. There is a simple observation for this family of functionals in a special case, where $\mu_k = \omega_0^k$, $k = 1, \dots, n$ for some Kähler metric ω_0 .

Proposition 6.1 For any $\phi \in \mathcal{H}_{\omega_0}$,

$$J_{\omega_0^n}(\phi) \ge J_{\omega_0^{n-1}}(\phi) \ge \dots \ge J_{\omega_0}(\phi) \ge \frac{1}{n+1} J(\phi)$$

Proof If suffices to prove $J_{\omega_0^{k+1}}(\phi) \geq J_{\omega_0^k}(\phi)$. However, this is almost obvious by the definition. Take the standard linear path $\phi_t = t\phi$, then

$$J_{\omega_{0}^{k}}(\phi) = \int_{0}^{1} dt \int_{M} \dot{\phi}_{t}(\omega_{0}^{k} - \omega_{\phi_{t}}^{k}) \omega_{\phi_{t}}^{n-k}$$

$$= \int_{0}^{1} dt \int_{M} \phi \ t(\omega_{0} - \omega_{\phi}) (\omega_{0}^{k-1} + \omega_{0}^{k-2} \omega_{t\phi} + \dots + \omega_{t\phi}^{k-1}) \omega_{t\phi}^{n-k}$$

$$= \int_{0}^{1} t \ dt \int_{M} \sqrt{-1} \partial \phi \wedge \bar{\partial} \phi \wedge (\omega_{0}^{k-1} \omega_{t\phi}^{n-k} + \omega_{0}^{k-2} \omega_{t\phi}^{n-k+1} + \dots + \omega_{t\phi}^{n-1})$$

Since each integrand is a positive term, $J_{\omega_0^{k+1}} \geq J_{\omega_0^k}$. And the last inequality $J_{\omega_0} \geq \frac{1}{n+1}J$ is well-known (cf. Bando-Mabuchi[3, Eq. 1.6.4]).

Similar to the case k = 1, there is also a moment map picture associated with this functional.

One can also define a new continuity path as

$$t(R_{\varphi} - \underline{R}) = (1 - t)(\frac{\mu_k \wedge \omega_{\varphi}^{n-k}}{\omega_{\varphi}^n} - c_k), \quad \forall t \in [0, 1]$$

This is the Euler-Lagrange equation of the twisted K-energy functional

$$E_{\mu_k,t} = tE + (1-t)J_{\mu_k}.$$

Then, one can formulate twisted K-stability and other geometric notions similarly. It is straightforward to prove the following

Theorem 6.2 For any closed positive $\chi > 0$, the twisted K-energy functional $E_{\mu_k,t}$ is convex along $C^{1,1}$ geodesic segment for any $k = 1, 2, \dots, n$.

Following proof in Section 3, we have

Theorem 6.3 For any k > 0, the twisted cscK metric equation is open for any $t \in [0,1)$.

Theorem 6.4 If $(M, [\omega])$ is K-semistable and K-stable for J_{μ_k} , then the twisted trip structure $(M, [\omega], \mu_k, t)$ is K-stable for $t \in [0, 1)$.

Moreover, we can ask question parallel to those in previous sections. To avoid redundancy, we only list the following conjecture

Conjecture 6.5 There exists a twisted cscK metric if and only if the corresponding twisted K-energy functional is proper in terms of geodesic distance.

Here we give a proof of the necessary part for k = n and $t \in (0, 1)$.

Proof Suppose ω_{φ} is a twisted cscK metric for $\mu > 0$ an positive volume form and $t_0 \in (0, 1)$, then there exists a small $\delta > 0$ such that the twisted cscK metric equation can be solved for any $t' \in (t_0, t_0 + 2\delta) \subset (0, 1)$. Fix $t' = t_0 + \delta$. Then,

$$t'E + (1 - t')J_{\mu}$$
.

is bounded from below.

Firstly, by Yau's solution of Calabi's Volume form Conjecture, there exists a unique Kähler metric $\omega_0 \in [\omega]$ such that

$$\mu = c_n \omega_0^n$$

which implies $J_{\mu}=c_nJ_{\omega_0^n}\geq \frac{c_n}{n+1}J$ by Prop. 6.1.

Now

$$\begin{array}{rcl} t_0 E + (1-t_0) J_\mu & = & \frac{t_0}{t'} (t' E + (1-t_0) \frac{t'}{t_0} J_\mu) \\ & = & \frac{t_0}{t'} (t' E + (1-t') J_\mu) + \frac{t_0}{t'} ((1-t_0) \frac{t'}{t_0} - (1-t')) J_\mu) \\ & = & \frac{t_0}{t'} (t' E + (1-t') J_\mu) + \frac{t_0}{t'} (\frac{t'}{t_0} - 1) J_\mu \\ & = & \frac{t_0}{t'} (t' E + (1-t') J_\mu) + \frac{t'-t_0}{t'} J_\mu \\ & \geq & \frac{t_0}{t'} (t' E + (1-t') J_\mu) + \frac{\delta}{\delta + t_0} J_\mu. \end{array}$$

Now the first term is bounded from below and the second is proper, thus the twisted K-energy $t_0E + (1-t_0)J_{\mu}$ is proper in terms of the geodesic distance.

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